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ORBITAL EJECTION REQUIREMENTS FOR RE-ENTRY AT CONSTANT ANGLE

β

5.93

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I. ABSTRACT

Based on simplified two body assumptions, a general analysis is presented of ballistic trajectories which result in a specified flight path angle at a given altitude. The minimum required velocity is obtained starting with zero velocity and from a circular orbit. The range is derived for initial intercepts at the specified altitude. An example of the results is presented in graphical form for an initial altitude of 300 nm and final altitudes of 450,000 ft. and 1028 nm.

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a = constant defined in Equation (4)

b = constant defined in Equation (4)

e = eccentricity

r = radial distance from center of attracting body

R = dimensionless distance r_o/r

v = velocity

V = dimensionless velocity v/v_c

ΔV = velocity impulse required staring from a circular orbit

α = dimensionless radial velocity $(V_o \sin \gamma_o)$

β = dimensionless horizontal velocity $(V_o \cos \gamma_o)$

γ = flight path angle with respect to the horizonal

δ = velocity (ΔV) angle with respect to the horizonal

θ = range central angle

λ_1 = Lagrange Multiplier

ϕ = true anomaly

SUBSCRIPTS

c = circular

d = down

i = inward

I = intercept

o = initial condition, outward

p = perigee

u = up

V. VELOCITY REQUIREMENTS

The velocity of a body in an orbit can be specified by a magnitude and a flight path angle, herein taken as the angle with respect to the local horizon. If, at the initial point, the dimensionless velocity components are represented by α (radial component) and β (horizontal component), the initial required velocity starting with zero velocity can be expressed as in Equation (1). The required velocity starting from a circular orbit ($\alpha = 0$, $\beta = 1$) is specified by Equation (2) (See Figure 1).

$$V_o = (\alpha^2 + \beta^2)^{1/2} \qquad (1)$$

$$\gamma_o = \tan^{-1} \frac{\alpha}{\beta}$$

$$\Delta V = \left[\alpha^2 + (1-\beta)^2\right]^{1/2} \qquad (2)$$

$$\delta = \tan^{-1} \frac{\alpha}{1-\beta}$$

The flight path angle at the desired altitude shall be referred to as the intercept angle (γ_I), and is derived from simplified two body assumptions.

$$\gamma_I = \tan^{-1} \frac{(-R^2\beta^2 + 2R - 2 + \alpha^2 + \beta^2)^{1/2}}{R\beta} \qquad (3)$$

where $R = \frac{r_o}{r_I}$ = initial distance/intercept distance .

Solving for the radial component of velocity gives

$$\alpha = (\beta^2 a + b)^{1/2} \qquad (4)$$

where

$$a = \frac{R^2}{\cos^2 \gamma_I} - 1$$

$$b = 2(1-R)$$

Using Equations (1), (2) and (4), β can be expressed as functions of γ_o, Equation (5), or as a function of δ, Equation 6.

$$\beta = \left(\frac{b}{\tan^2 \gamma_o - a} \right)^{\frac{1}{2}} \qquad (5)$$

$$\beta = \frac{\tan^2 \delta \pm \left([a+b] \tan^2 \delta - ab \right)^{\frac{1}{2}}}{\tan^2 \delta - a} \qquad (6)$$

$$+\ \delta > 90^\circ ; \ -\ \delta < 90^\circ ; \ \delta = 90^\circ : \beta = 1$$

VI. MINIMUM ENERGY REQUIREMENTS

It is apparent from substituting Equation (4) into Equation (1) that the minimization of the velocity (V_0) for a specified R and γ_I occurs for the minimum permissible β. For outward intercepts ($R < 1$, $b > 0$), Equation (5) is minimized for $\gamma_o = 90^\circ$ ($\beta = 0$); for inward intercepts ($R > 1$, $b < 0$), $\gamma_o = 0$ ($\alpha = 0$). The corresponding velocities are found by combining Equations (1) and (4).

$$R < 1 : \quad V_o = b^{\frac{1}{2}}$$
$$R > 1 : \quad V_o = \left(- \frac{b}{a} \right)^{\frac{1}{2}} \qquad (7)$$

Minimization of the required velocity (ΔV) starting from a circular orbit is accomplished by minimizing Equation (2) subject to the constraint on α (Equation (4)). Using Lagrange Multipliers, the following three equations are obtained:

$$\alpha^2 - a\beta^2 - b = 0$$
$$\frac{\alpha}{\left[\alpha^2 + (1-\beta)^2 \right]^{\frac{1}{2}}} + 2\lambda_1 \alpha = 0 \qquad (8)$$
$$\frac{1-\beta}{\left[\alpha^2 + (1-\beta)^2 \right]^{\frac{1}{2}}} + 2\lambda_1 a\beta = 0$$

Using Equation (2) and assuming $\alpha \neq 0$, the solution is given by

$$\beta = \cos^2\gamma_I / R^2$$
$$\Delta V = (3 - 2R - \cos^2\gamma_I / R^2)^{1/2} \qquad (9)$$
$$\delta = \cos^{-1}\{[\Delta V^2 - 2(1-R)] / \Delta V\}$$

For $\alpha = 0$, the solution to Equation (8) is given by Equation (10), with ΔV and δ obtained from Equation (2).

$$\beta = (-b/a)^{1/2}$$
$$\Delta V = |1 - (-b/a)^{1/2}| \qquad (10)$$
$$R > 1: \quad \delta = 0^\circ$$
$$R < 1: \quad \delta = 180^\circ$$

The region where the solution for $\alpha = 0$ is applicable is determined by setting Equation (4) equal to zero, resulting in Equation (11).

$$\beta = [1 \pm (9 - 8R)^{1/2}]/2$$
$$\gamma_{I_{a,b}} = \cos^{-1}\{R[1 \pm (9 - 8R)^{1/2}]/2^{1/2}\} \qquad (11)$$
$$R > 1: \quad \gamma_{I_a} < \gamma_I < \gamma_{I_b}$$
$$R < 1: \quad \gamma_{I_a} < \gamma_I < 90^\circ$$

The solution to minimizing V_0 (Equation (7)) is equivalent to minimizing ΔV (Equation (10)) ($\Delta V = 1 - V_0$ when $\alpha = \gamma_0 = \delta = 0^\circ$) for $R > 1.125$ and when $1 < R < 1.125$ with γ_I outside the range given in Equation (11).

VII. RANGE CONSIDERATIONS

The above analysis results in those velocity vectors which produce intercept at the specified intercept angle and altitude. Each vector, in general, will result in intercept at two different ranges depending on whether or not the velocity is applied upward or downward. This is due to the shape of the orbit being independent of the direction of the

initial velocity angle.

The intercept point is oriented relative to the initial point by the range angle (θ) and to the perigee point by the true anomaly (ϕ) which in turn is oriented relative to the initial point by the perigee angle. There are two variables to consider in writing the range equations: 1) whether the desired intercept is to be at a greater or lesser altitude, and 2) whether the impulse is applied upward or downward. Figure 2 shows the four possibilities, from which the following ranges of angles are inferred for the cases restricted to initial intercept, and the angles (measured from the last fictitious perigee passage in the rotation direction) restricted to the principal values.

Outward Intercepts

Upward application of velocity

$0° < \theta_p < 180°$

$0° < \phi < 180°$

$\theta_{o_u} = \phi - \theta_p$

Downward application of velocity

$180° < \theta_p < 360°$

$360° < \phi < 540°$

$\theta_{o_d} = \phi + \theta_p$

Inward Intercepts

Upward application of velocity

$0° < \theta_p < 180°$

$180° < \phi < 360°$

$\theta_{i_u} = 2\pi - \phi + \theta_p$

Downward application of velocity

$180° < \theta_p < 360°$

$180° < \phi < 360°$

$$\theta_{i_d} = \theta_p - \phi$$

The true anomaly and perigee angles are given by

$$\theta_p = \cos^{-1}\left[(\beta^2-1)/e\right]$$

$$\phi = \cos^{-1}\left[(\beta^2 R-1)/e\right] \qquad (12)$$

where $e = \left[\alpha^2\beta^2 + (1-\beta^2)\right]^{1/2}$

The range equations can then be derived.

$$\theta_{i_d} = \cos^{-1}\frac{\beta^2-1}{e} - \cos^{-1}\frac{\beta^2 R-1}{e}$$

$$\theta_{i_u} = 2\pi - \cos^{-1}\frac{\beta^2-1}{e} - \cos^{-1}\frac{\beta^2 R-1}{e} \qquad (13)$$

$$\theta_{o_d} = \cos^{-1}\frac{\beta^2 R-1}{e} + \cos^{-1}\frac{\beta^2-1}{e}$$

$$\theta_{o_u} = \cos^{-1}\frac{\beta^2 R-1}{e} - \cos^{-1}\frac{\beta^2-1}{e}$$

For inward intercepts ($R > 1$) the maximum applicable δ and accompaning minimum range is obtained from the geometry for the limiting case of a straight line transfer with infinite velocity.

$$\delta_{max.} = \cos^{-1}\left(-\frac{\cos\gamma_I}{R}\right)$$

$$\theta_{min.} = \cos^{-1}\left(\frac{\cos\gamma_I}{R}\right) - \gamma_I \qquad (14)$$

The maximum range (maintaining the original orbital direction) occurs for an upward application of velocity of escape magnitude ($V_o = 2^{\frac{1}{2}}$), resulting in an eccentricity of one. Equating the normal velocity components from Equation (4) and Equation (1) yields the required velocity (ΔV) and the range.

$$\beta = (2/R)^{\frac{1}{2}}\cos\gamma_I$$

$$\Delta V = \left[3 - 2^{3/2}\cos\gamma_I / R^{\frac{1}{2}}\right]^{\frac{1}{2}} \qquad (15)$$

$$\theta_{max.} = 2\pi - \cos^{-1}(2\cos^2\gamma_I/R - 1) - \cos^{-1}(2\cos^2\gamma_I - 1)$$

For outward intercepts ($R < 1$) there is a minimum δ (maintaining the original orbital direction) which corresponds to a vertical trajectory ($\gamma_o = 90^\circ$, $\beta = 0$) that just reaches the desired altitude. The velocity (V_o) required is given by Equation (7), which results in δ min.

$$\delta_{min.} = \tan^{-1}\left[2(R-1)\right]^{1/2} \qquad (16)$$

The maximum applicable angle (δ) and corresponding ranges (the range differs for upward and downward applications of velocity) are determined from the geometry of a straight line transfer.

$$\delta_{max.} = \cos^{-1}(-\cos\gamma_I / R)$$
$$\theta = \gamma_I \pm \cos^{-1}(\cos\gamma_I / R) \qquad (17)$$
$$\text{where } \gamma_I > \cos^{-1} R$$

VIII DISCUSSION

For the case of starting from a circular orbit, the region where a horizontal application of velocity does not yield a minimum ΔV is shown in Figures 3 and 4. An $R > 1.125$ always requires a horizontal application of velocity, an $R \sim 1$ generally requires a non-horizontal application of velocity, and an $R \sim 0$ generally requires a horizontal application.

An example of an altitude descent from 150 n. mi. to 450,000 ft. is shown in Figures 5 and 6. The required $\Delta V - \delta$ figure shows that the required velocity increases rapidly with increasing intercept angle. A considerable reduction in velocity is obtained by ejection at a $\delta \neq 0$

in some cases. An intercept angle of 30 degrees, for example, results in a velocity requirement difference of 4,700 ft/sec between $\delta = 0$ degrees and $\delta = 55$ degrees. In general, the velocity is relatively insensitive to changes in δ at the minimum energy points.

The range (Figure 5) is double valued due to upward and downward applications of impulse producing different ranges. Small intercept angles allow almost complete coverage of the lower altitude; large intercept angles a very restricted range. An intercept angle of 80 degrees, for example, covers the central angle range from 0.2 to 42 degrees. The minimum energy points show a rapid decrease in range as intercept angle increases from zero degrees. When the minimum energy points are first obtained for $\delta \neq 0$, the range becomes double valued and shows both an increase and decrease in the range, with the range eventually decreasing to approximately zero for intercept angles close to 90 degrees.

Figure 7 shows the required $\Delta V - \delta$ for intercepts occurring at an altitude of 1028 n. mi. starting from a 150 n. mi. circular orbit. The minimum application angle (δ) is 32 degrees which corresponds to a vertical application ($\gamma_o = 90$) which just reaches the required altitude at apogee. Any injection angle can be obtained at this point by slightly changing the initial conditions from a vertical ejection, causing the orbit to arc over just above the designated altitude.

For small intercept angles ($\gamma_I < 23^o$) the minimum energy transfer occurs when $\delta = 180$ degrees. For $\gamma_I > 23$ degrees, δ rapidly decreases to 32 degrees ($\gamma_I = 90^o$).

The range is shown in Figure 8 and is based on an infinitestimal sized attracting body. All the curves pass through the ranges of 0 and 360 degrees at a velocity of 30,000 ft/sec (corresponding to a vertical trajectory). The range of 360 degrees is a somewhat fictitious one, for a vertical downward ejection would in actuality pass through the attracting body, and in the case of an infinitestimal size mass, the body would pass through the center and continue on a straight line (due to conservation of linear momentum.) A slight deviation from a vertical downward V_1 would result in the body being "wipped" around the center (because of conservation of angular momentum), resulting in a range close to 360 degrees.

Intercept angles greater than 36.6 degrees do not allow complete range coverage. A γ_I of 50 degrees, for example, cannot be attained for ranges between 13 and 87 degrees.

Straight line trajectories are illustrated in Figures 9 and 10.

IX. CONCLUSIONS

1. For inward intercepts with R > 1.125, the minimum energy transfer trajectory starting with zero velocity or from a circular orbit is obtained by a horizontal application of velocity.

2. For inward intercepts with R < 1.125, there are two ranges of intercept angles (one starting from zero degrees and the other ending at 90 degrees) where conclusion one is true. When the intercept angle lies between the two ranges, minimum energy transfers starting from a point with zero velocity are obtained with a horizontal application of velocity; minimum energy transfers starting from a circular orbit

are obtained with a non-horizontal application of velocity.

3. For outward intercepts, minimum energy transfers starting from a point with zero velocity are obtained by a vertical ejection resulting in a range of zero degrees. Minimum energy transfers starting from a circular orbit are obtained by ejecting horizontally for a limited range of intercept angles starting at zero degrees, outside of this range the angle of velocity application varies between 180 degrees and some angle less than 90 degrees.

4. Inward intercepts result in a limited range coverage, excluding a range angle near zero degrees and near 360 degrees.

5. Outward intercepts, for a range of intercept angles starting at zero degrees, allow complete coverage. For a range of intercept angles extending to 90 degrees, there is a range of range angles which cannot be attained.

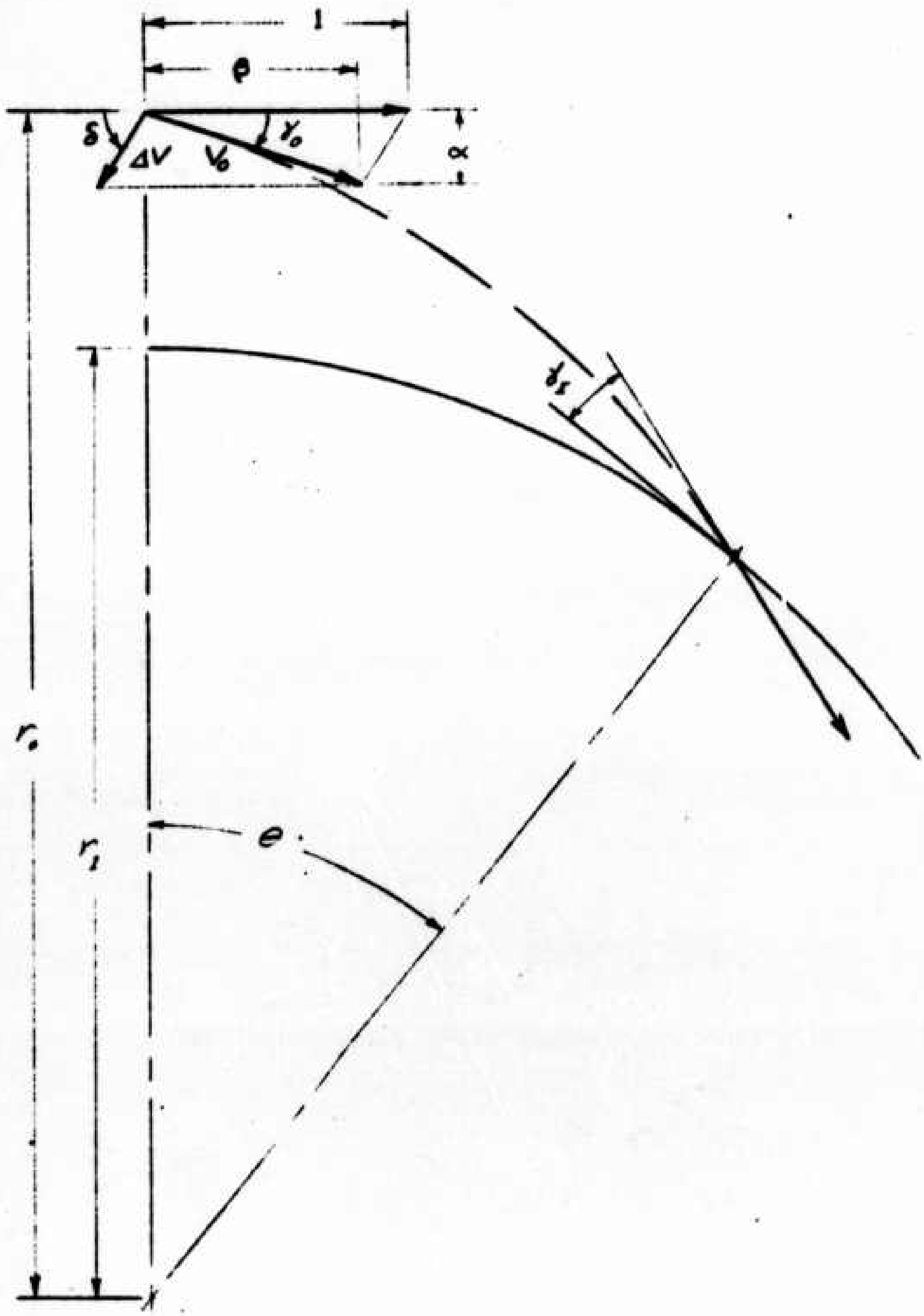


Figure 1. Illustration of Nomenclature

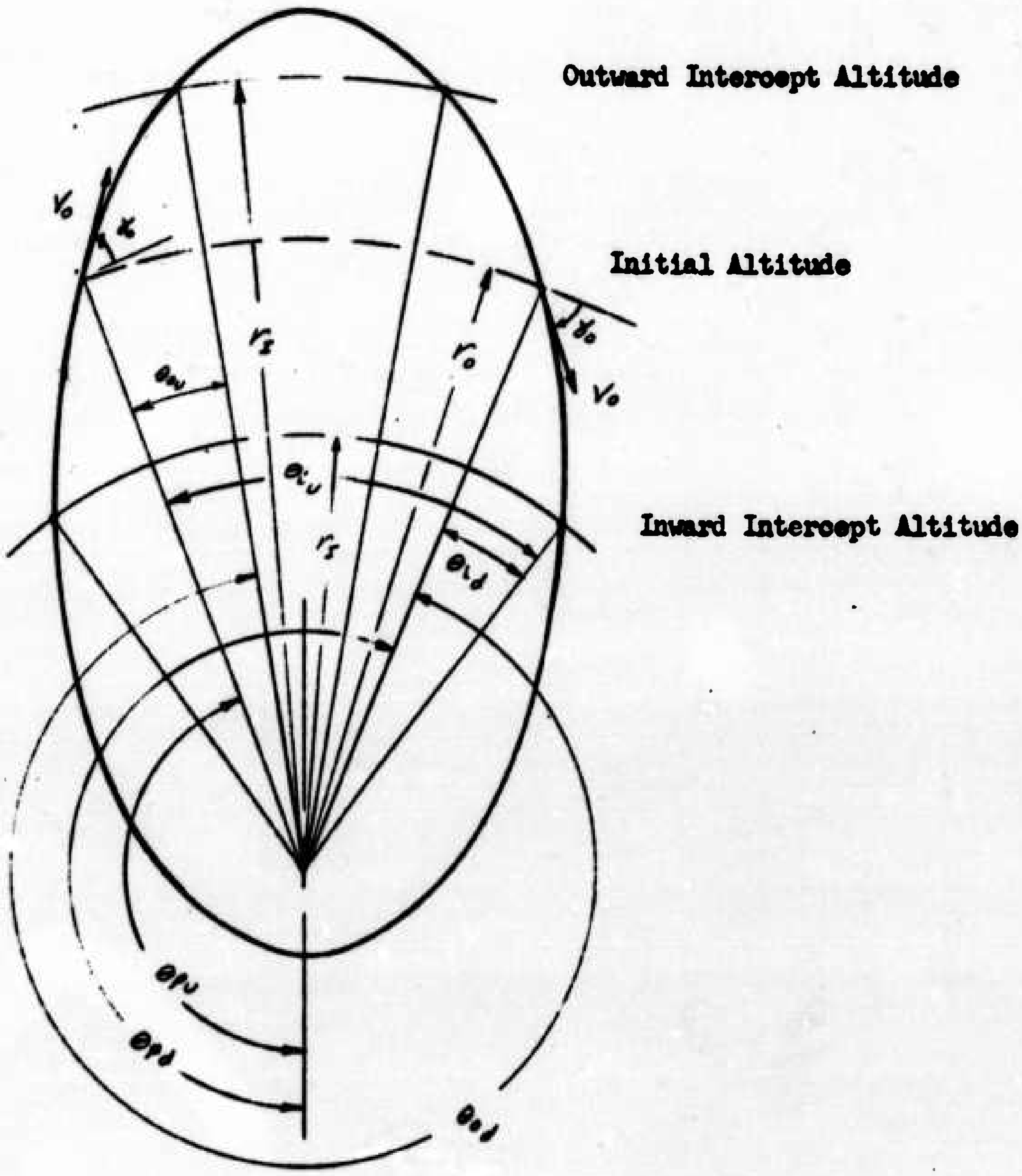


Figure 2. ILLUSTRATION OF RANGE AND PERIGEE ANGLES

Figure 3. Region where Minimum Energy Transfers for a Constant Intercept Angle are Obtained with Impulse Application Angles of Zero Degrees, E > 1

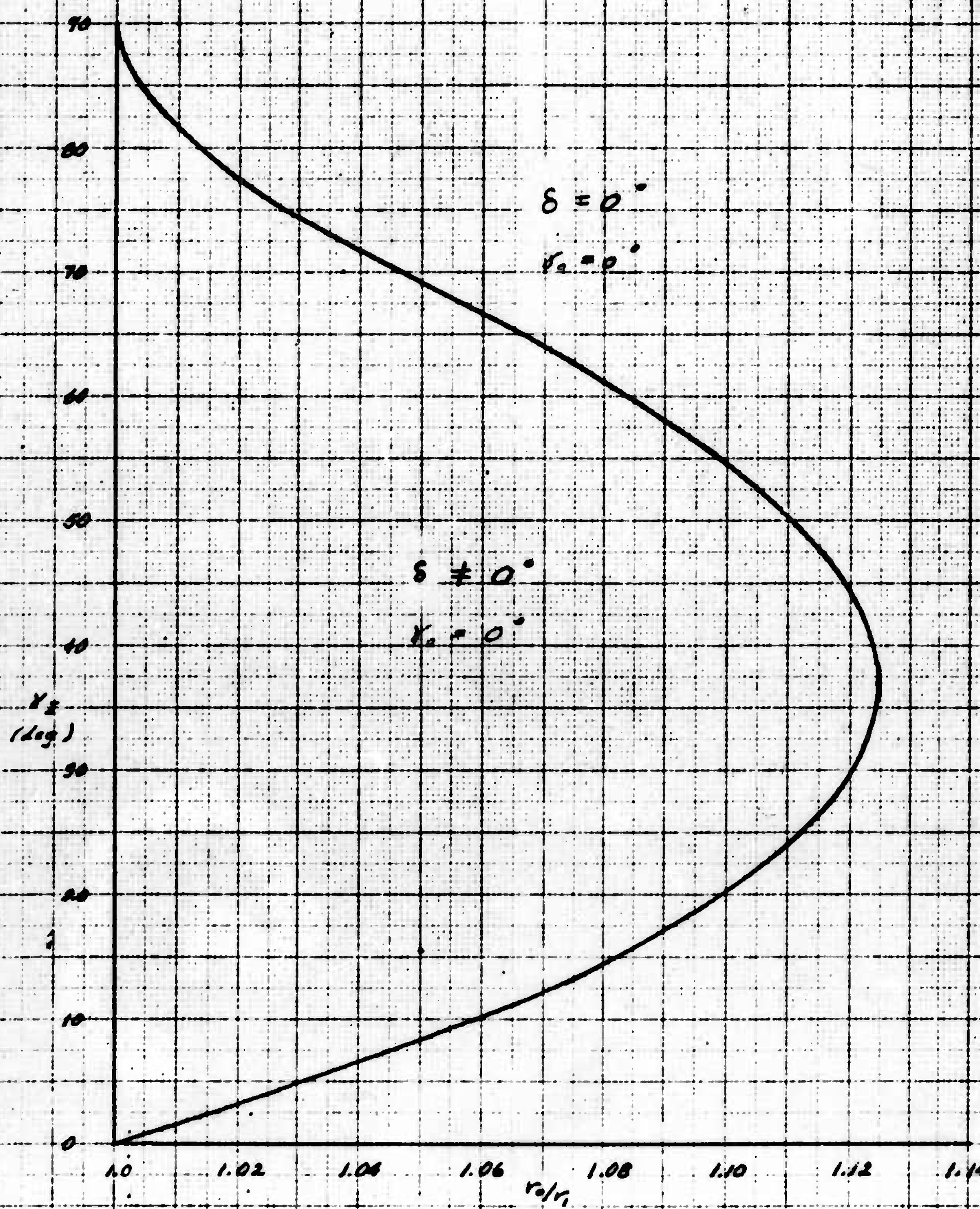


Figure 4. Region where Minimum Energy Transfers for a Constant Intercept Angle are Obtained with constant Impulse Application Angles R < 1

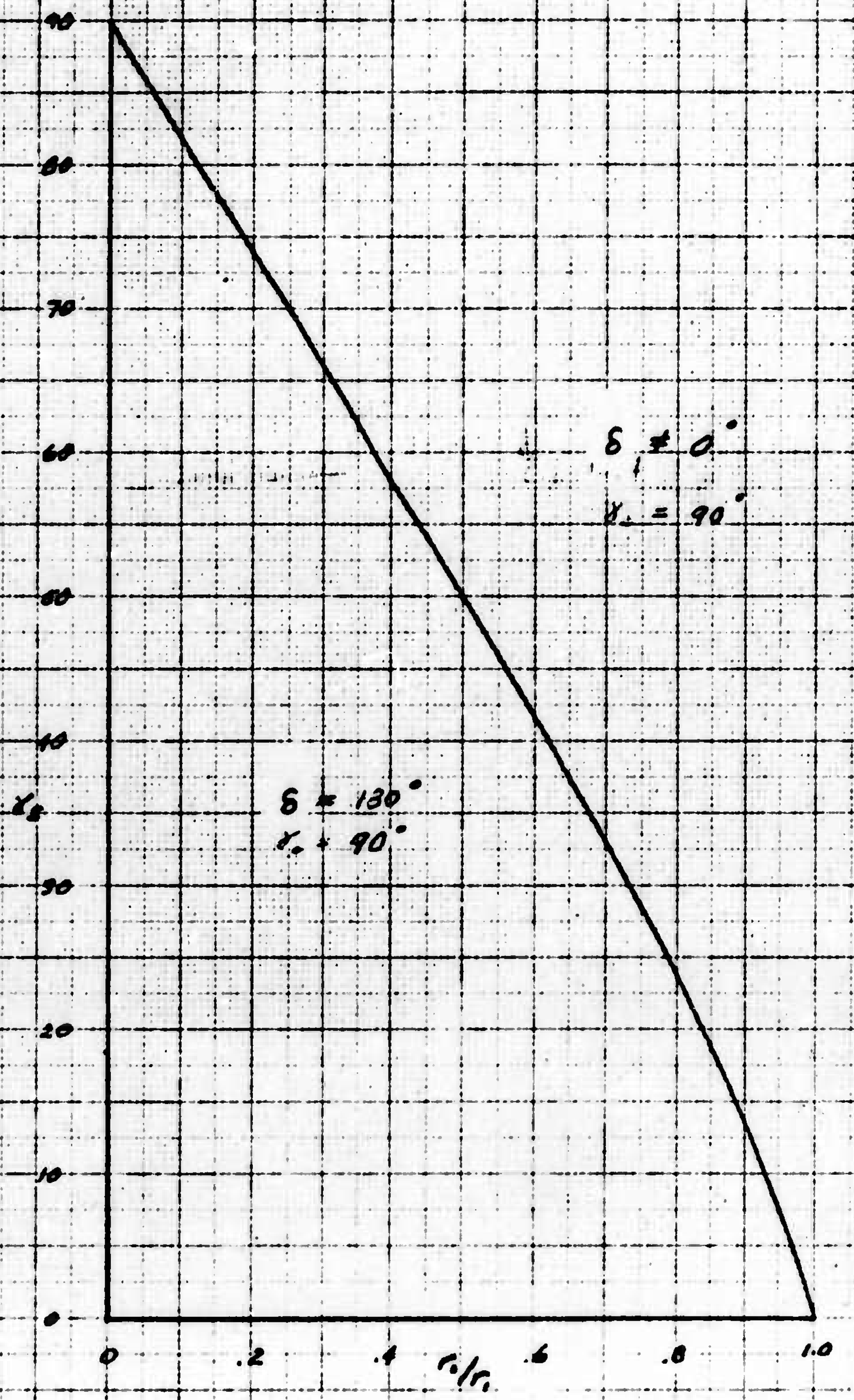


FIGURE 5 . IMPULSE APPLICATION ANGLE FOR DESCENT FROM 150 NM to 450,000 FT.

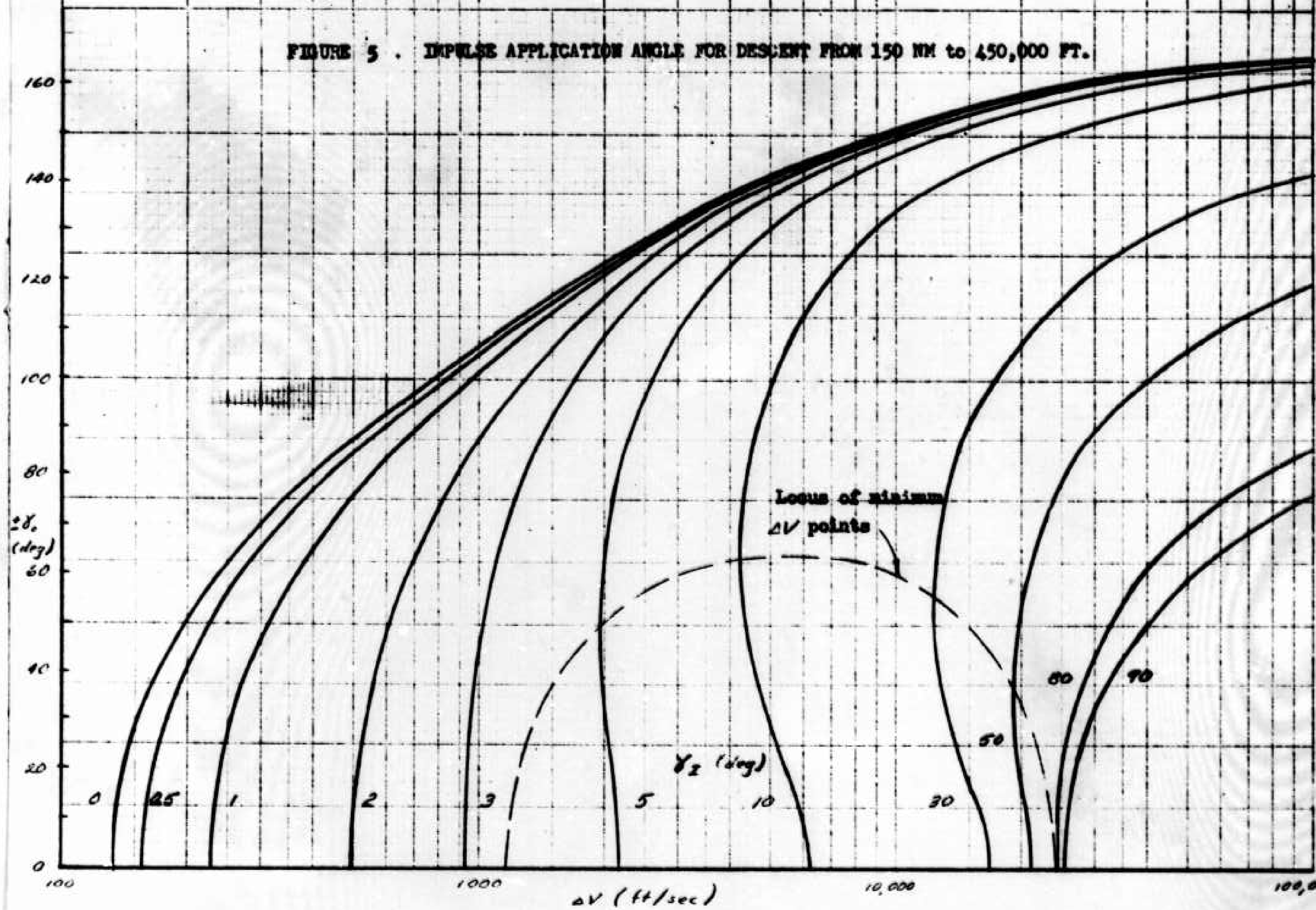
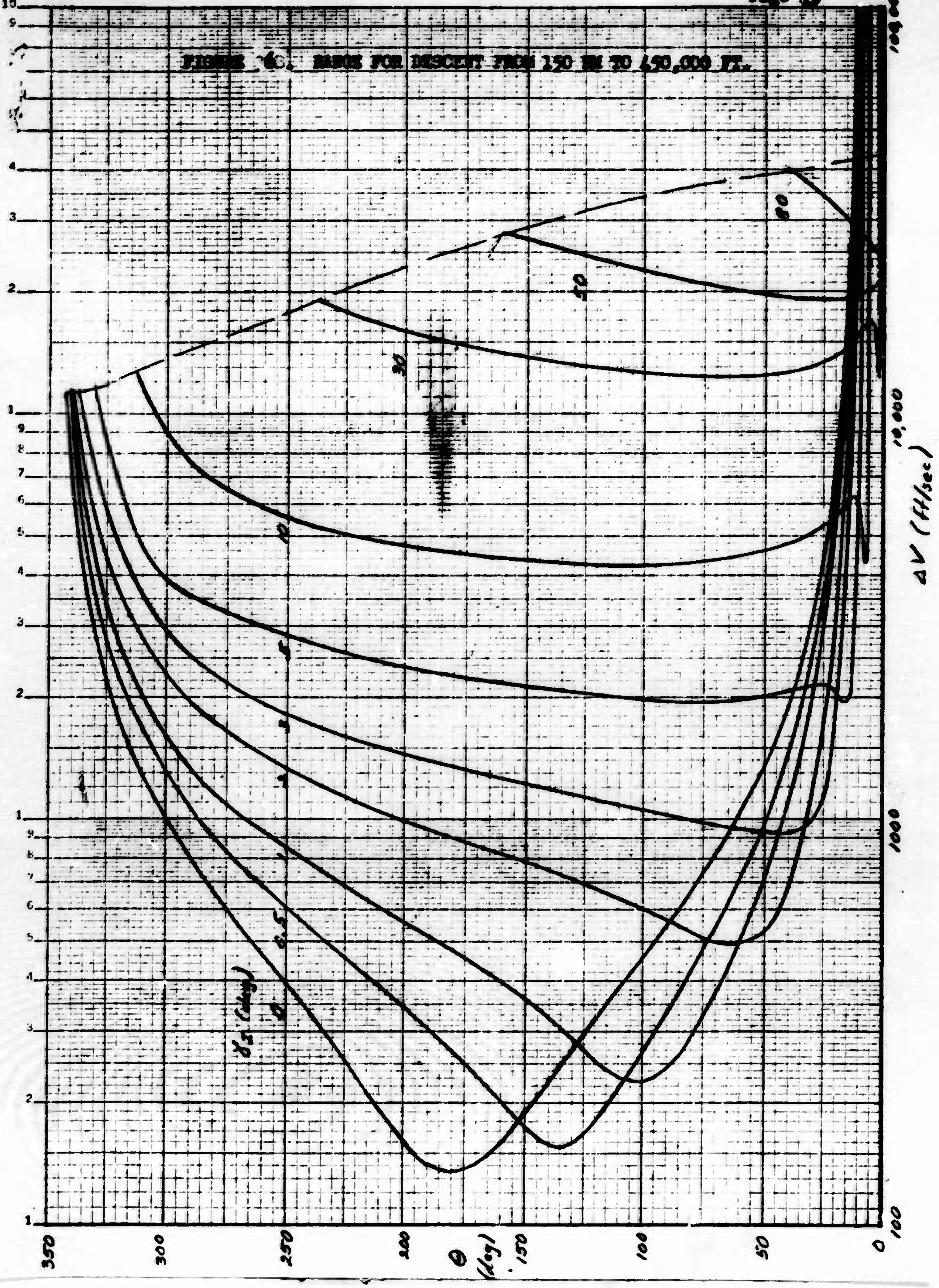


FIGURE 6C. RANGE FOR DESCENT FROM 150 NM TO 450,000 FT.



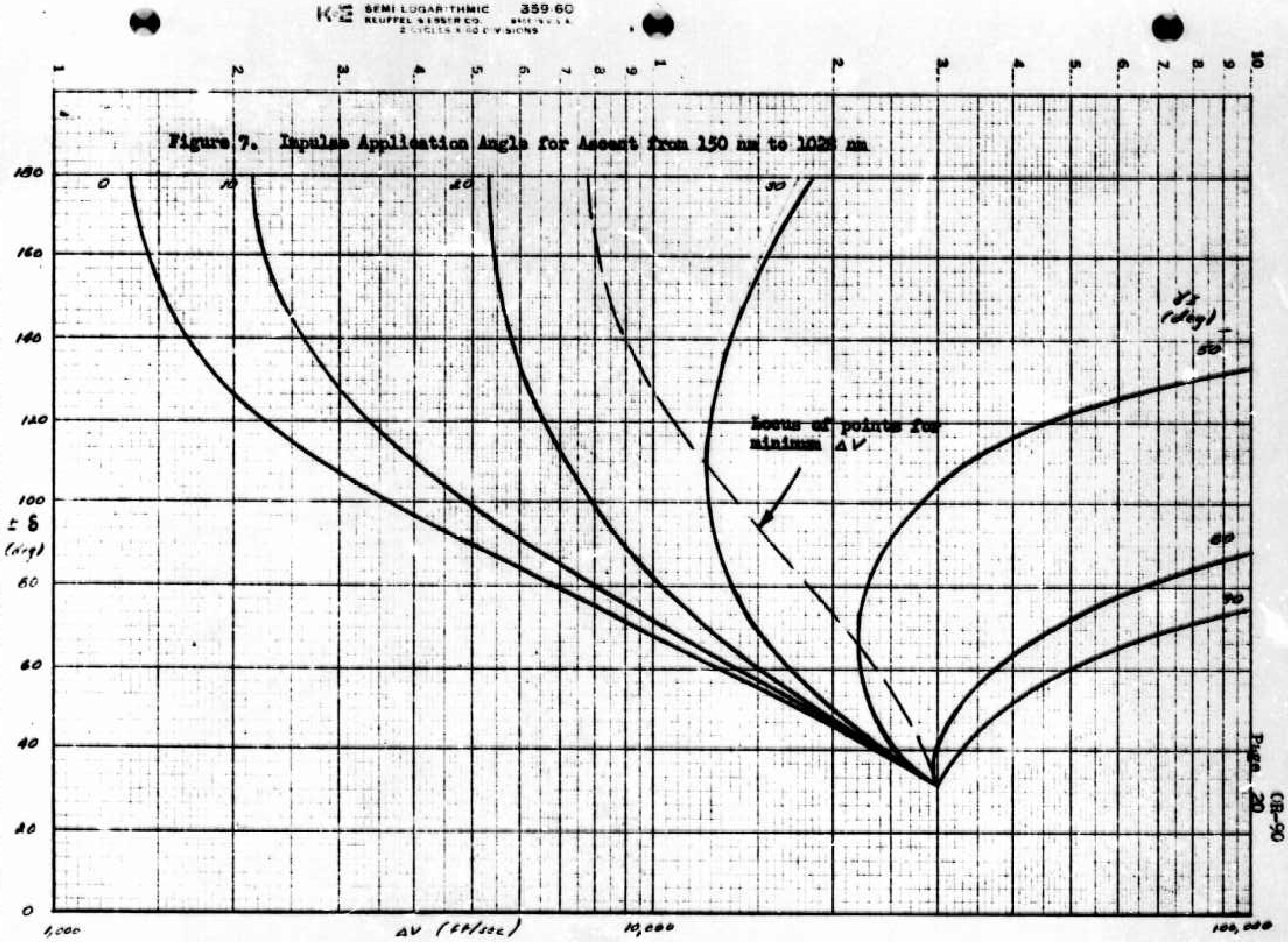


Figure 7. Impulse Application Angle for Ascent from 150 nm to 1028 nm

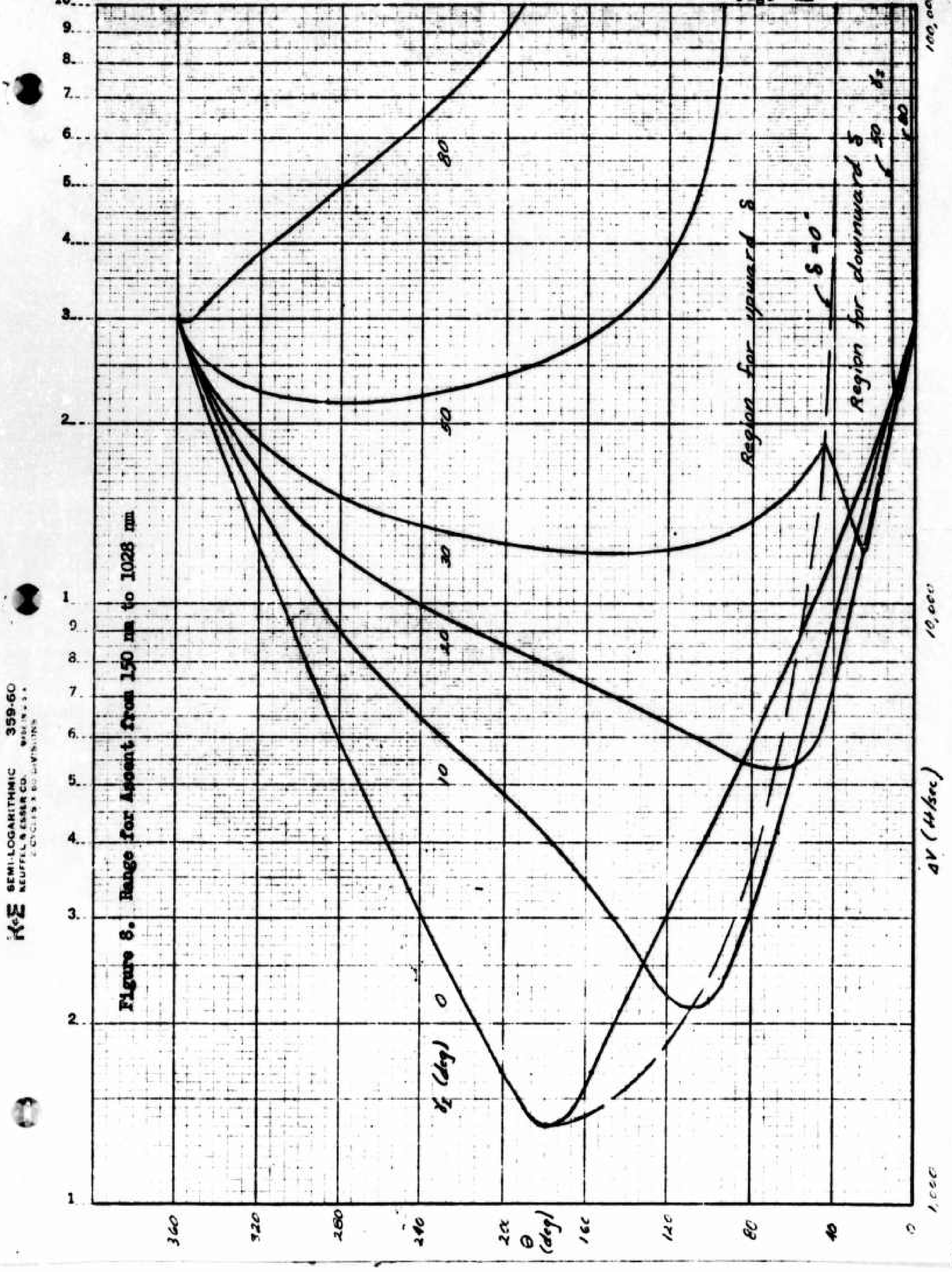
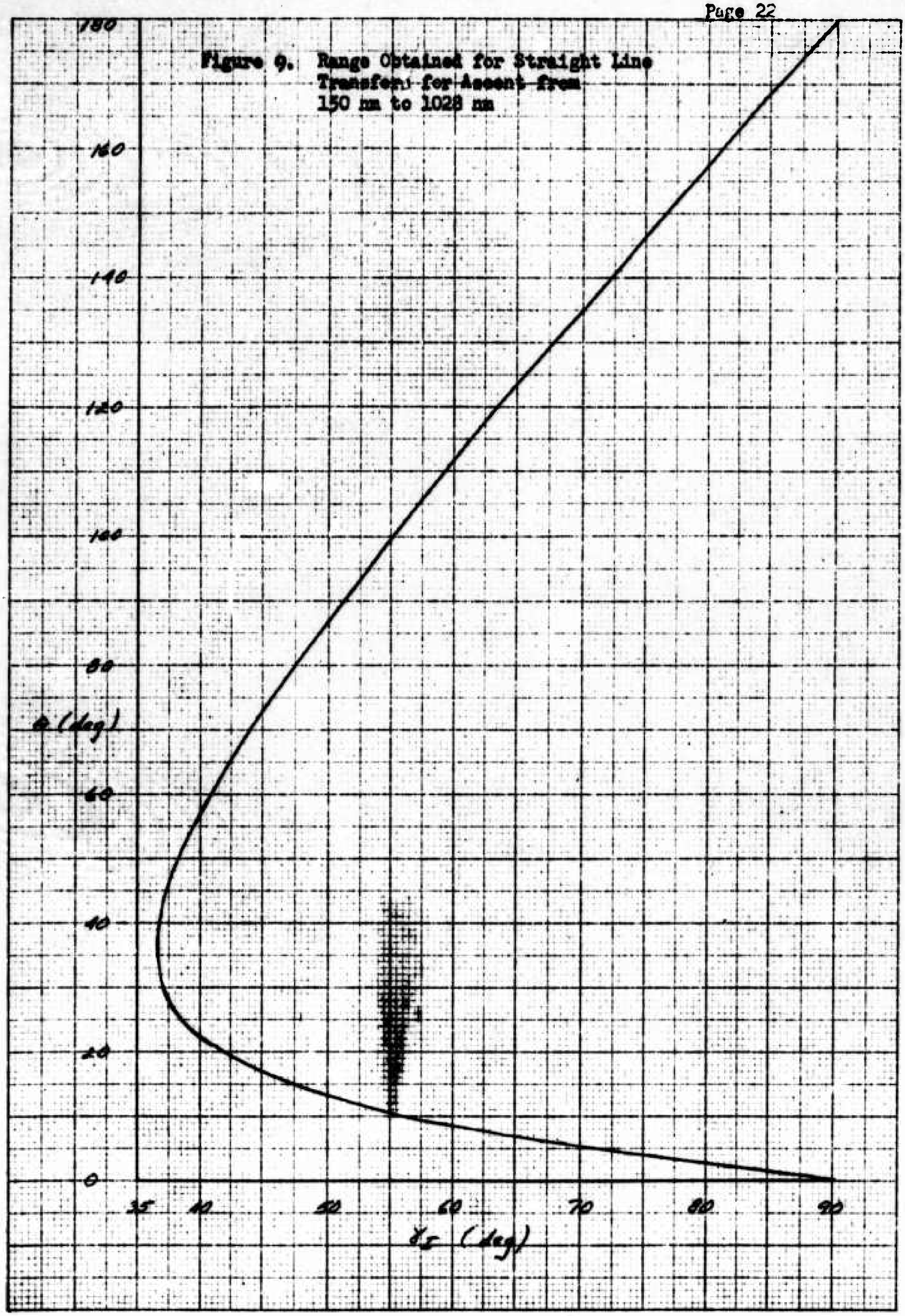


Figure 8. Range for Ascent from 150 nm to 1028 nm

Figure 9. Range Obtained for Straight Line Transfer for Ascent from 150 nm to 1028 nm



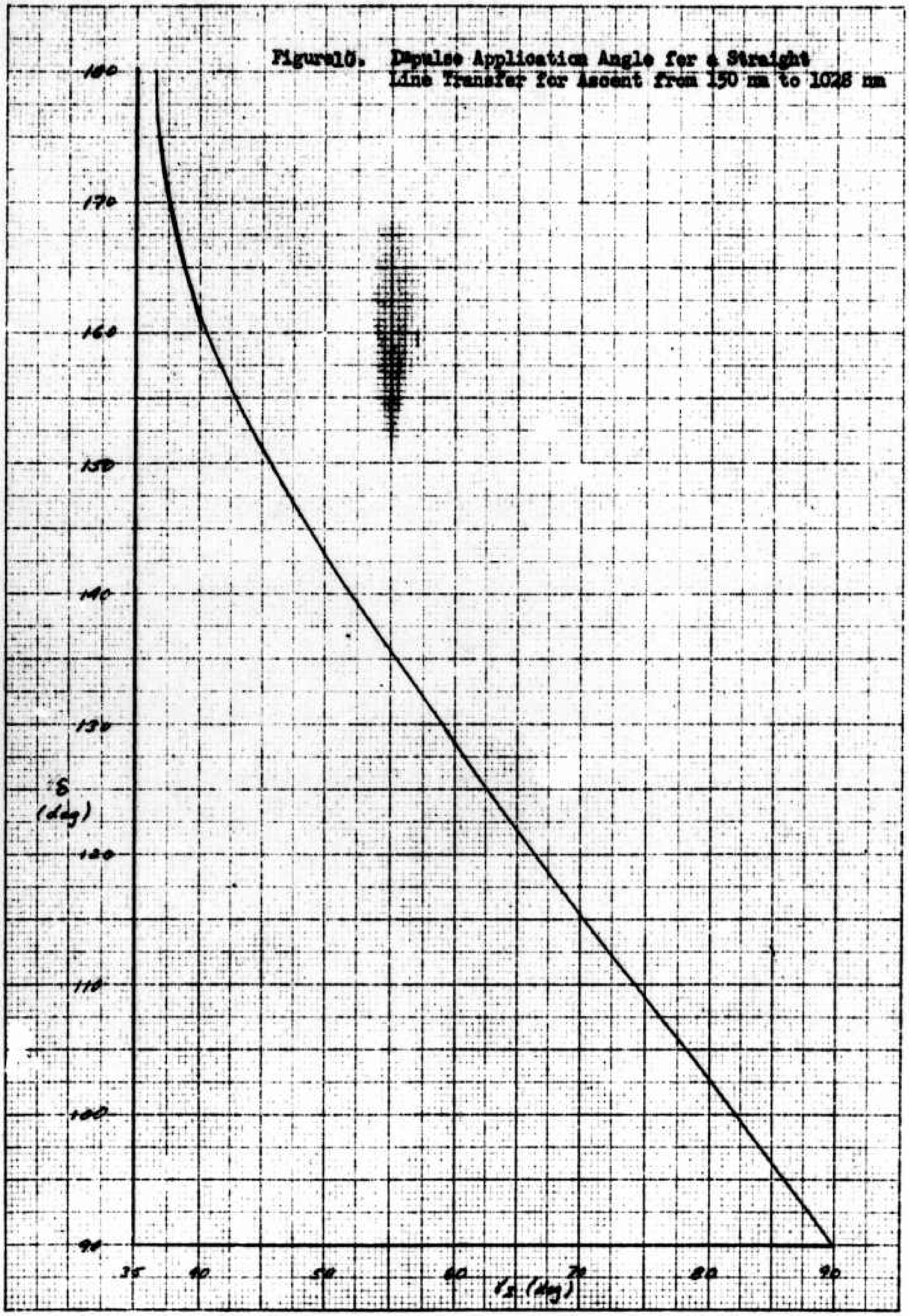


Figure 10. Impulse Application Angle for a Straight Line Transfer for Ascent from 150 nm to 1028 nm